

Coordinates of $R[x, y]$: Constructions and classifications.

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Abstract. Let R be a PID. We construct and classify all coordinates of $R[x, y]$ of the form $p_2y + Q_2(p_1x + Q_1(y))$ with $p_1, p_2 \in \text{qt}(R)$ and $Q_1, Q_2 \in \text{qt}(R)[y]$. From this construction (with $R = K[z]$) we obtain non tame automorphisms σ of $K[x, y, z]$ (where K is a field of characteristic 0) such that the sub-group generated by σ and the affine automorphisms contains all tame automorphisms.

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1 Introduction

1) Let K be a field. Due to the famous Jung-van der Kulk theorem (cf. [15] and [16]), the group of all automorphisms of the K -algebra $K[x, y]$ is generated by the sub-groups of affine automorphisms and triangular automorphisms. Moreover, this group is the amalgamated product of these two sub-groups along their intersection (cf. [12]). This result allows us both to construct all automorphisms of $K[x, y]$ (by composition) and to classify them in terms of the length or the polydegree (see [13] and [14]).

2) There exist two classical ways to construct automorphisms of the K -algebra $K[x, y, z]$. The first one is to compose affine automorphisms and triangular automorphisms (we obtain the so called tame automorphisms). Sheshtakov and Umirbaev have proved that, if K is a field of characteristic 0, we do not obtain all automorphisms of $K[x, y, z]$ in this way (cf. [21]). The second way consists to extend automorphisms of the $K[z]$ -algebra $K[z][x, y]$ to obtain automorphisms of $K[x, y, z]$ fixing z (called z -automorphisms). This idea is developed in [8] (see also [17] §9.4). We do not know whether all automorphisms of $K[x, y, z]$ may be obtained by composing z -automorphisms and affine ones.

3) In this context, it is natural to study the automorphisms of the R -algebra $R[x, y]$ thinking that R is a PID, a UFD, a domain or simply a general ring. When R is a domain, an automorphism of $R[x, y]$ is roughly defined by one

of his component (cf. Corollary 2). This is the reason why we focus our attention on coordinates of $R[x, y]$.

4) In section 2, we introduce some classical notations and we recall well-known theorems: Nagata (see Theorem 1), Russell-Sathaye (see Theorem 2) and Shestakov-Umirbaev (see Theorem 3).

5) In section 3, we give the construction of automorphisms with one component of the form $d^{-1}\{q_2y + Q_2(q_1dx + Q_1(y))\} \in R[x, y]$ (cf. Theorem 6).

6) In section 4, we develop the first elements of a theory of classification of these automorphisms. We distinguish which are tame and which come from Russell-Sathaye construction (cf. Theorem 8). We prove that, if R is a PID, all polynomials of the form $p_2y + Q_2(p_1x + Q_1(y)) \in R[x, y]$ with $p_1, p_2 \in \text{qt}(R)$ and $Q_1, Q_2 \in \text{qt}(R)[y]$ can be written with the form considered in Theorem 6 (cf. Theorem 7).

7) There are many motivations to construct such automorphisms of $R[x, y]$:
- To construct non tame automorphisms of $K[z][x, y]$ (see for example [8]) which give non tame automorphisms of $K[x, y, z]$ using Shestakov-Umirbaev theorem (cf. [21]).

- To construct families of automorphisms of $\mathbb{C}[x, y]$ with generic length 3 to study the closure of the set automorphisms of $\mathbb{C}[x, y]$ with a fixed polydegree (see [10]).

- To give a criterion to check if there exists an automorphism of $R[x, y]$ sending $p_1x + Q_1(y)$ to $p_2x + Q_2(y)$ where $p_1, p_2 \in R \setminus \{0\}$ and $Q_1, Q_2 \in R[y]$ (see section 5). This question is linked with the work of Poloni (cf. [19]) about the classification of Danielewski hypersurfaces.

- To obtain non tame automorphisms σ of $K[x, y, z]$ (where K is a field of characteristic 0) such that the sub-group generated by σ and the affine automorphisms contains all tame automorphisms (see Section 6).

2 Preliminaries

Notation 1 Let R be a commutative ring.

a) We denote by R^* the multiplicative group of units of R and by R^\times the set of non zero-divisors of R (when R is a domain, we have $R^\times = R \setminus \{0\}$). We denote by $\text{qt}(R) = (R^\times)^{-1}R$ the total quotient ring of R (when R is a domain, $\text{qt}(R)$ is the field of fractions of R). We denote by R^\times/R^* the quotient of R^\times by the equivalence relation \sim defined by $r \sim s$ if and only if there exists $u \in R^*$ such that $r = us$, for all $r, s \in R^\times$. We fix a subset $\mathcal{U}(R)$ of R^\times such that, for all $r \in R^\times$, there exists a unique element $w_R(r) \in \mathcal{U}(R)$ such that $r \sim w_R(r)$. For example, we can take for $\mathcal{U}(K[z])$ the set of unitary polynomials with $w_{K[z]}(P(z)) = \frac{1}{\text{lt}(P(z))}P(z)$ (where $\text{lt}(P(z))$ is the leading

term of $P(z)$), for all $P(z) \in K[z]^\times$ and we can take for $\mathcal{U}(\mathbb{Z})$ the set $\mathbb{N} \setminus \{0\}$ with $w_{\mathbb{Z}}(n) = |n|$, for all $n \in \mathbb{Z}^\times$. We denote by $\text{Nil}(R)$ the ideal of nilpotent elements in R .

b) Let $f_1, \dots, f_n \in R[x_1, \dots, x_n]$ be polynomials, we denote by (f_1, \dots, f_n) the endomorphism σ of the R -algebra $R[x_1, \dots, x_n]$ defined by $\sigma(x_i) = f_i$, for all $i \in \{1, \dots, n\}$.

c) Let $f_1, f_2 \in R[x, y]$, we denote by $\det(J\sigma) = (\partial_x f_1)(\partial_y f_2) - (\partial_x f_2)(\partial_y f_1)$ the Jacobian determinant of the endomorphism (f_1, f_2) .

d) We denote by $\text{GA}_n(R)$ ($n \in \mathbb{N} \setminus \{0\}$) the automorphisms group of the R -algebra $R[x_1, \dots, x_n]$ (when $n = 1$, $x_1 = y$, when $n = 2$, $(x_1, x_2) = (x, y)$ and when $n = 3$, $(x_1, x_2, x_3) = (x, y, z)$).

e) We denote by $\pi = (y, x) \in \text{GA}_2(R)$ or $\pi = (y, x, z) \in \text{GA}_3(R)$ the automorphism exchanging x and y .

f) We denote by $\text{VA}_n(R) = \{F \in R[x_1, \dots, x_n]; \exists \sigma \in \text{GA}_n(R); \sigma(x_n) = F\}$. the set of R -coordinates (or R -variables) of $R[x_1, \dots, x_n]$.

g) If R is a domain, we use the following notations:

$\text{Aff}_n(R) = \{\sigma \in \text{GA}_n(R); \forall i \in \{1, \dots, n\}, \deg(\sigma(x_i)) = 1\}$ (for the affine automorphisms group),

$\text{BA}_n(R) = \{\sigma \in \text{GA}_n(R); \forall i \in \{1, \dots, n\}, \sigma(x_i) \in R^*x_i + R[x_{i+1}, \dots, x_n]\}$ (for the triangular automorphisms group) and

$\text{TA}_n(R) = \langle \text{Aff}_n(R), \text{BA}_n(R) \rangle$ (for the tame automorphisms group).

The following theorem is well-known and describes $\text{VA}_1(R)$ (cf. [18]).

Theorem 1 (Nagata, 1972) *Let $P \in R[y]$ be a polynomial. The following assumptions are equivalent:*

i) $P \in \text{VA}_1(R)$,

ii) *there exist $r \in R$, $u \in R^*$ and $N \in \text{Nil}(R[y])$ such that:*

$$P(y) = uy + r + N(y).$$

Remark

1) There exists an algorithm and even an explicit formula (see [12] Theorem 3.1.1 for the characteristic zero case and [1] Theorem 6.2 for the positive characteristic case) to compute the inverse of the automorphism $\sigma \in \text{GA}_1(R)$ defined by $\sigma(y) = P$. So we can say that $\text{VA}_1(R)$ is well understood for every commutative ring R .

2) It is very important, in Theorem 1, to consider a ring which is not a domain (since if R is a domain $\text{VA}_1(R)$ contains only affine polynomials). Nevertheless, when we study $\text{VA}_2(R)$, we often assume that R is a domain, a UFD (unique factorization domain) or even a PID (principal ideal domain) because the main applications are for $R = K[z]$, where K is field.

3) If R is a \mathbb{Q} -algebra, assumptions i) and ii) of Theorem 1 are equivalent to: iii) $P'(y) \in R[y]^*$.

The following corollaries of Theorem 1 are useful.

Corollary 1 *We have: $\text{VA}_2(R) + \text{Nil}(R[x, y]) = \text{VA}_2(R)$.*

Corollary 2 *Let $\sigma, \tau \in \text{GA}_2(R)$ be automorphisms such that $\sigma(y) = \tau(y)$. We set: $Y = \sigma(y) = \tau(y)$. We have: $\sigma(x) = u(Y)\tau(x) + r(Y) + N(x, y)$ with $r \in R[y]$, $u \in R[y]^*$ and $N \in \text{Nil}(R[x, y])$. If R is a domain, we have: $\sigma(x) = u\tau(x) + r(Y)$ with $r \in R[y]$ and $u \in R^*$.*

Remark This last corollary shows that in $R[x, y]$, where R is a domain, a coordinate is exactly the orbit of an automorphism under the action of the group of triangular automorphisms.

A first natural idea is to try to describe $\text{VA}_2(R)$ using $\text{VA}_1(R/I)$ for some (principal) ideals I of R .

Notation 2 Let I be an ideal of R . The canonical morphism $\phi_I : R \rightarrow R/I$ may be extended to a morphism from $R[x, y]$ to $(R/I)[x, y]$. We still denote this morphism ϕ_I . If $I = pR$ for some $p \in R$, we set: $\phi_p = \phi_{pR}$.

The second natural idea is to define subclasses of $R[x, y]$ and to try to describe the intersection between $\text{VA}_2(R)$ and each of these classes. The following definition come from [3]:

Definition 1 (Berson, 2002) Let $p_1, \dots, p_l \in R^\times$, and $Q_1, \dots, Q_l \in R[y]$. We define $F_l \in R[x, y]$ by induction on $l \in \mathbb{N} \setminus \{0\}$:

- 1) $F_1(x, y) = p_1x + Q_1(y)$,
- 2) $F_2(x, y) = p_2y + Q_2(p_1x + Q_1(y))$,
- 3) $F_l(x, y) = p_lF_{l-2} + Q_l(F_{l-1})$, for all $l \geq 2$. We set (for $l \in \mathbb{N} \setminus \{0\}$):

$$\mathcal{B}^l(R) = \{F_l(x, y) ; p_1, \dots, p_l \in R^\times, Q_1, \dots, Q_l \in R[y]\} \subset R[x, y]$$

and $\mathcal{B}^0(R) = \{p_0y + q_0 ; p_0 \in R^\times, q_0 \in R\}$.

Definition 2 (Rational classes) Let $l \in \mathbb{N}$ be an integer. We define:

$$\mathcal{R}^l(R) = \mathcal{B}^l(\text{qt}(R)) \cap R[x, y].$$

Remark We assume that R is a domain.

1) Let $l \in \mathbb{N}$ be an integer. A polynomial $F \in R[x, y]$ belongs to $\mathcal{R}^l(R)$ if and only if there exist $\tau_1, \dots, \tau_{l+1} \in \text{BA}_2(\text{qt}(R))$ such that $F = \tau_1 \pi \dots \tau_l \pi \tau_{l+1}(y)$ and we can assume that $\tau_i(y) = y$ for $i \in \{1, \dots, l\}$ (see [14]).

2) By Jung-van der Kulk theorem, we have $\text{GA}_2(\text{qt}(R)) = \text{TA}_2(\text{qt}(R))$. Using Bruhat decomposition in $\text{Gl}_2(\text{qt}(R))$ we deduce that:

$$\text{VA}_2(R) = \bigcup_{l \in \mathbb{N}} \mathcal{R}^l(R) \cap \text{VA}_2(R).$$

3) The first component of the Nagata automorphism (see point 5 in Remark after Theorem 2) is in $\mathcal{R}^2(R) \cap \text{VA}_2(R)$ but is not in $\bigcup_{l \in \mathbb{N}} \mathcal{B}^l(R)$ (see Proposition 1).

4) We have $\mathcal{B}^0(R) = \mathcal{R}^0(R)$ and $\mathcal{B}^1(R) = \mathcal{R}^1(R)$ but $\mathcal{B}^2(R) \subsetneq \mathcal{R}^2(R)$.

The description of $\mathcal{B}^1(R) \cap \text{VA}_2(R) = \mathcal{R}^1(R) \cap \text{VA}_2(R)$ has been done by Russell and Sathaye (cf. [20]):

Theorem 2 (Russell, Sathaye, 1976) *Let $p_1 \in R^\times$ be a non zero-divisor and let $Q_1 \in R[y]$ be a polynomial. We set $F(x, y) = p_1 x + Q_1(y)$. The following assumptions are equivalent:*

- i) $F(x, y) \in \text{VA}_2(R)$,
- ii) $\phi_{p_1}(F(x, y)) \in \text{VA}_2(R/p_1 R)$,
- iii) $\phi_{p_1}(Q_1(y)) \in \text{VA}_1(R/p_1 R)$.

Remark

- 1) Theorem 2 is a particular case of Theorem 5 ($Q_2(y) = y$).
- 2) Theorem 2 is true for all $p_1 \in R$ (including zero-divisors) as shown by Berson (see Theorem 1.2.6. in [4]).
- 3) If we assume that R is a domain and suppose iii). For all $u \in R^*$ and $Q_2 \in R[y]$ such that $Q_2(Q_1(y)) = y \bmod p_1 R[y]$. We have:

$$\sigma = ((up_1)^{-1}(y - Q_2(F(x, y))), F(x, y)) \in \text{GA}_2(R)$$

and (by Corollary 2) every $\sigma \in \text{GA}_2(R)$ such that $\sigma(y) = F(x, y)$ has this form.

4) With the notations of the previous point, we have: $\sigma \in \text{TA}_2(R)$ if and only if there exist $a \in R^*$ and $b \in R$ such that $Q_1(y) = ax + b$ modulo $p_1 R[y]$ (see for example [9]).

5) The very classical example is the Nagata automorphism ($R = K[z]$ where K is a field, $p_1 = z^2$, $Q_1(y) = y + zy^2$, $Q_2(y) = y - zy^2$ and $u = -1$).

Let K be a field. The groups $\text{GA}_2(K[z])$ and $\{\sigma \in \text{GA}_3(K); \sigma(z) = z\}$ are canonically isomorphic (by the map $(f_1, f_2) \rightarrow (f_1, f_2, z)$). In this way, we can consider $\text{TA}_2(K[z])$ as a sub-group of $\text{GA}_3(K)$.

Theorem 3 (Shestakov, Umirbaev, 2004)

Let K be a field of characteristic 0.

$$\{\sigma \in \text{GA}_3(K); \sigma(z) = z\} \cap \text{TA}_3(K) = \text{TA}_2(K[z]).$$

This theorem (see [21]) is very strong because it's easy to check if a z -automorphisms is in $\text{TA}_2(K[z])$ (see [9], see also [14] for an algorithm). In particular, we know, since [18], that the Nagata automorphism is not in $\text{TA}_2(K[z])$ and Theorem 3 implies is not in $\text{TA}_3(K)$. An even stronger result is obtain in [22] where a conjecture from [18] is solved:

Theorem 4 (Umirbaev, Yu 2004)

Let K be a field of characteristic 0. Let $\sigma \in \text{GA}_2(K[z]) \setminus \text{TA}_2(K[z])$ be a wild automorphism of $K[z][x, y]$ then there exists no tame automorphism $\tau \in \text{TA}_3(K)$ such that $\tau(y) = \sigma(y)$ (we say that $\sigma(y)$ is a wild coordinate of $K[x, y, z]$).

3 Length 2 constructions.

The description of $F(x, y) = p_2y + Q_2(p_1x + Q_2(y)) \in \mathcal{B}^2(R) \cap \text{VA}_2(R)$ begins in [11] with the case $p_2 = 1$ and in [9] for the case $p_1R + p_2R = R$. The case $p_2 = 1$ also appear independently in [8] (see also [17] §9.4) in the case $R = K[z]$ where K is a field of characteristic 0. A complete characterization is given in Theorem 5. In Theorem 6 we study $\mathcal{R}^2(R) \cap \text{VA}_2(R)$ (which strictly contains $\mathcal{B}^2(R) \cap \text{VA}_2(R)$). Theorem 5 is a particular case of Theorem 6 ($d = 1, p_1 = q_1$ and $p_2 = q_2$).

Theorem 5 *Let $p_1, p_2 \in R^\times$ be non zero-divisors and let $Q_1, Q_2 \in R[y]$ be polynomials. We set: $F(x, y) = p_2y + Q_2(p_1x + Q_1(y)) \in R[x, y]$. The following assumptions are equivalent:*

- i) $F(x, y) \in \text{VA}_2(R)$,
- ii) a) $\phi_{p_1}(F(x, y)) \in \text{VA}_2(R/p_1R)$ and b) $\phi_{p_2}(F(x, y)) \in \text{VA}_2(R/p_2R)$,
- iii) a) $\phi_{p_1}(p_2y + Q_2(Q_1(y))) \in \text{VA}_1(R/p_1R)$ and b) $\phi_{p_2}(Q_2(y)) \in \text{VA}_1(R/p_2R)$.

Example 1 We assume that $R = K[z]$, where K is a field. We set: $p_1 = z^2(z - 1)$, $p_2 = z$, $Q_1(y) = y + zy^2$ and $Q_2(y) = (z - 1)(y + zy^2)$. The polynomial $F(x, y) = p_2y + Q_2(p_1x + Q_1(y))$ is a coordinate by Theorem 5.

We have $p_1R + p_2R \neq R$. Moreover $F(x, y)$ is not a coordinate of length "1 + 1" (*i. e.* is not a component of an automorphism composed by two automorphisms constructed in Theorem 2, see Definition 5.3).

Theorem 6 *Let $d, q_1, q_2 \in R^\times$ be non zero-divisors such that $dR + q_2R = R$ and let $Q_1, Q_2 \in R[y]$ be polynomials such that $\phi_d(q_2y + Q_2(Q_1(y))) = 0$. We set: $F(x, y) = d^{-1}\{q_2y + Q_2(q_1dx + Q_1(y))\} \in R[x, y]$. The following assumptions are equivalent:*

- i) $F(x, y) \in \text{VA}_2(R)$,
- ii) a) $\phi_{q_1}(F(x, y)) \in \text{VA}_2(R/q_1R)$ and b) $\phi_{q_2}(F(x, y)) \in \text{VA}_2(R/q_2R)$.
- iii) a) $\phi_{q_1}(F(0, y)) \in \text{VA}_1(R/q_1R)$ and b) $\phi_{q_2}(Q_2(y)) \in \text{VA}_1(R/q_2R)$.

Before proving Theorem 6, we recall the following three classical lemmas. Lemma 1 is obvious, Lemma 2 is a consequence of Lemma 1.11 in [3], and Lemma 3 is a consequence of Lemma 1.1.8 in [12] (which is a corollary of the formal inverse function theorem).

Lemma 1 *Let $F \in R[x, y]$ and let I be an ideal of R . If $F \in \text{VA}_2(R)$ then $\phi_I(F) \in \text{VA}_2(R/I)$.*

Lemma 2 *Let $Q \in R[y]$ and $H \in R[x, y]$. The following assumptions are equivalent:*

- i) $Q(H(x, y)) \in \text{VA}_2(R)$,
 - ii) $Q(y) \in \text{VA}_1(R)$ and $H(x, y) \in \text{VA}_2(R)$,
- In particular, $\text{VA}_2(R) \cap R[y] = \text{VA}_1(R)$.*

Lemma 3 *Let σ be an endomorphism of the R -algebra $R[x, y]$. We have $\sigma \in \text{GA}_2(R)$ if and only if the following two assumptions are fulfilled:*

- i) $\det(J\sigma)(0) \in R^*$,
- ii) $\sigma \in \text{GA}_2(\text{qt}(R))$.

Proof (of Theorem 6).

i) \Rightarrow ii). This follows from Lemma 1.

ii) \Rightarrow iii). In this part of the proof, we use Lemma 2.

a) Since $\phi_{q_1}(F(x, y)) \in \text{VA}_2(R/q_1R)$, we have $\phi_{q_1}(F(0, y)) \in \text{VA}_2(R/q_1R)$ and $\phi_{q_1}(F(0, y)) \in \text{VA}_1(R/q_1R)$ using Lemma 2.

b) (This part of the proof is the only one where we use the hypothesis $dR + q_2R = R$). Since $\phi_{q_2}(F(x, y)) \in \text{VA}_2(R/q_2R)$ and since d is an invertible element modulo q_2 , we have $\phi_{q_2}(Q_2(q_1dx + Q_1(y))) \in \text{VA}_1(R/q_2R)$ and $\phi_{q_2}(Q_2(y)) \in \text{VA}_1(R/q_2R)$ using Lemma 2.

iii) \Rightarrow i). In this part of the proof, we use Lemma 3.

By b), there exist $S, U \in R[y]$ such that $S(Q_2(y)) = y + q_2U(y)$ (1). There

exists $V \in R[x, y]$ such that $S(q_2y + x) - S(x) = q_2V(x, y)$.

Changing x to $Q_2(q_1dx + Q_1(y))$ in the previous equation, we have:

$$\begin{aligned} S(dF(x, y)) - S(Q_2(q_1dx + Q_1(y))) &= q_2V(Q_2(q_1dx + Q_1(y)), y) \\ &= q_2W(y) \bmod q_1q_2R[x, y] \quad (2), \end{aligned}$$

where $W(y) = V(Q_2(Q_1(y)), y) \in R[y]$.

By a), there exists $T \in R[y]$ such that $T(F(0, y)) = y \bmod q_1R[y]$. We have:

$$T(F(x, y)) = T(F(0, y)) = y \bmod q_1R[y] \quad (3).$$

We set: $Q_3(y) = S(dy) - q_2\{U(Q_1(T(y))) + W(T(y))\}$.

Modulo $q_1q_2R[x, y]$, we have:

$$\begin{aligned} Q_3(F(x, y)) &= S(dF(x, y)) - q_2\{U(Q_1(y)) + W(y)\} && \text{(by (3))} \\ &= S(Q_2(q_1dx + Q_1(y))) - q_2U(Q_1(y)) && \text{(by (2))}, \\ &= q_1dx + Q_1(y) + q_2(U(q_1dx + Q_1(y)) - U(Q_1(y))) && \text{(by (1))}, \\ &= q_1dx + Q_1(y). \end{aligned}$$

Finally $q_1dx + Q_1(y) - Q_3(F(x, y)) = 0 \bmod q_1q_2R[x, y] \quad (4)$.

We consider the following endomorphisms of $\text{qt}(R)[x, y]$: $\tau_1 = (q_1dx + Q_1(y), y)$, $\tau_2 = (d^{-1}\{q_2x + Q_2(y)\}, y)$, $\tau_3 = ((q_1q_2)^{-1}(x - Q_3(y)), y)$, (we recall that $\pi = (y, x)$). We compute:

$$\sigma = \tau_1\pi\tau_2\pi\tau_3 = ((q_1q_2)^{-1}\{q_1dx + Q_1(y) - Q_3(F(x, y))\}, F(x, y)).$$

By (4), σ is an endomorphism of $R[x, y]$. By the chain rule, we have: $\det(J\sigma) = \det(J\tau_1)\det(J\tau_2)\det(J\tau_3) = q_1dd^{-1}q_2(q_1q_2)^{-1} = 1$. Using Lemma 3, we conclude that $\sigma \in \text{GA}_2(R)$ and $F(x, y) \in \text{VA}_2(R)$.

Remark We use the notations of Theorem 6.

1) We have: $\sigma^{-1} = \tau_3^{-1}\pi\tau_2^{-1}\pi\tau_1^{-1}$, hence:

$$\sigma^{-1} = ((q_1d)^{-1}\{q_1q_2(x + Q_3(y)) - Q_1(G(x, y))\}, G(x, y)),$$

where $G(x, y) = q_2^{-1}dy - q_2^{-1}Q_2(q_1q_2x + Q_3(y))$. It is not easy to prove directly (without using Lemma 3) that the first component of σ^{-1} is a polynomial in particular if $p_1R + p_2R \neq R$.

2) We set: $a = Q_2'(0)$ and $N(y) = Q_2(y) - ay$. We have: $Q_2(y) = ay + N(y)$. If we do not assume $dR + q_2R = R$ (which is equivalent to $\phi_{q_2}(d) \in (R/q_2R)^*$) but the weaker assumption $\phi_{q_2}(d) \in (R/q_2R)^\times$ then the condition ii) b) in Theorem 6 is equivalent to the following one:

iii)' b) $\phi_{q_2}(N) \in \text{Nil}(R/q_2R[y])$ and $\phi_{q_2R+aq_1R}(F(0, y)) \in \text{VA}_1(R/q_2R+aq_1R)$.

But we don't know whether iii) a) and iii)' b) imply i).

We justify this equivalence. Since $\phi_{q_2}(d) \in (R/q_2R)^\times$, we can consider the localization in $\phi_{q_2}(d)$ of the ring R/q_2R :

$$R/q_2R[d^{-1}] = \{x \in \text{qt}(R/q_2R) ; \exists n \in \mathbb{N}, \phi_{q_2}(d)^n x \in R/q_2R\}.$$

If $\phi_{q_2}(F(x, y)) \in \text{VA}_2(R/q_2R)$ then $Q_2(dq_1x + Q_1(y)) \in \text{VA}_2(R/q_2R[d^{-1}])$.

By Lemma 2, we deduce that $Q_2(y) \in \text{VA}_1(R/q_2R[d^{-1}])$. This implies that $\phi_{q_2}(N)$ is nilpotent in $R/q_2R[d^{-1}][y]$ and then in $R/q_2R[y]$.

Now, we assume that $\phi_{q_2}(N) \in \text{Nil}(R/q_2R[y])$.

We have: $\phi_{q_2}(d^{-1}(N(Q_1(y)) - N(dq_1x + Q_1(y)))) \in \text{Nil}(R/q_2R[x, y])$ and by Corollary 1:

$$\phi_{q_2}(F(x, y)) \in \text{VA}_2(R/q_2R)$$

$$\Leftrightarrow \phi_{q_2}(aq_1x + d^{-1}(q_2y + aQ_1(y) + N(Q_1(y)))) \in \text{VA}_2(R/q_2R)$$

$$\Leftrightarrow \phi_{q_2R+aq_1R}(F(0, y)) \in \text{VA}_1(R/q_2R + aq_1R).$$

The last \Leftrightarrow is justified by Berson's improvement of Russell-Sathaye Theorem (see Remark 2 of Theorem 2).

3) If R is a \mathbb{Q} -algebra, Theorem 6 is a consequence of the following deep result on locally nilpotent derivation due to Daigle and Freudenburg (for the case R a UFD, see [7]), Bhatwadekar and Dutta (for the case R normal noetherian domain, see [6]), Berson, van den Essen and Maubach (for the general case, see [5]):

Let R be a \mathbb{Q} -algebra and let $F \in R[x, y]$ be a polynomial. We have: $F \in \text{VA}_2(R)$ if and only if $F \in \text{VA}_2(\text{qt}(R))$ and $(\partial_x F)R + (\partial_y F)R = R[x, y]$. Actually, we prove that *iii*) implies $(\partial_x F)R + (\partial_y F)R = R[x, y]$. We set: $I = (\partial_x F)R + (\partial_y F)R$, $I_1 = I + q_1R[x, y]$ and $I_2 = I + q_3R[x, y]$. By *iii*) a), using the remark 3) of Theorem 1, we have: $\phi_{q_1}(F(0, y)') \in R/q_1R[y]^*$. We deduce that $0 = \phi_{I_1}(\partial_y F(x, y)) = \phi_{I_1}(F(0, y)') \in R/I_1[x, y]^*$. Hence $I_1 = R$ and $\phi_I(q_1) \in (R/I)^*$ (1). By *iii*) b), using Remark 3) of Theorem 1, we have: $\phi_{q_2}(Q_2'(y)) \in R/q_2R[y]^*$. Using (1) we obtain: $0 = \phi_{I_2}(\partial_x F(x, y)) = \phi_{I_2}(q_1)\phi_{I_2}(Q_2'(q_1dx + Q_1(y))) \in R/I_2[x, y]^*$. Hence $I_2 = R$ and $\phi_I(q_1) \in R/I^*$ (2). Finally, using (1) and (2), we have: $0 = \phi_I(dq_1\partial_y F(x, y) - Q_1'(y)\partial_x F(x, y)) = \phi_I(q_1q_2) \in R/I[x, y]^*$ and $I = R$.

Example 2 Let R be a PID. We give a general family of examples. Let $d, p_1, p_2 \in R^\times$ be such that $\gcd(d, p_1) = \gcd(d, p_2) = \gcd(p_1, p_2) = 1$ and let $u, v \in R$ be such that $du + q_1v = 1$. Let $Q_3, Q_4 \in R[y]$ be two polynomials such that $\phi_{q_2}(Q_4(y)) \in \text{Nil}(R/q_2R)$. We consider: $Q_1(y) = y + dQ_3(y)$, $Q_2(y) = q_1\{(d - q_2v)y + dQ_4(y)\}$ and $F(x, y) = d^{-1}\{q_2y + Q_2(q_1dx + Q_1(y))\}$. We have: $\phi_d(q_2y + Q_2(Q_1(y))) = \phi_d(q_2(1 - q_1v)y) = 0$. In one hand, we have: $\phi_{q_1}(F(0, y)) = \phi_{q_1}(d^{-1}q_2y) \in \text{VA}_1(R/q_1R)$ and, on the other hand, we have: $\phi_{q_2}(Q_2(y)) = \phi_{q_2}(q_1dy + dQ_4(y)) \in \text{VA}_1(R/q_2R)$. The assumption *iii*) of Theorem 6 is fulfilled and we deduce that $F(x, y) \in \text{VA}_2(R)$. Let's now give explicit examples:

Consider the ring $K[z]$, where K is a field, and take: $d = z^2$, $q_1 = (z - 1)^2$, $q_2 = (z - 2)^2$, $Q_1(y) = y + z^2y^2$ and

$$Q_2(y) = (z - 1)^2\{-2z^3 + 8z^2 - 4z - 4\}y + z^2(z - 2)y^2\}.$$

Consider the ring $R = \mathbb{Z}$ of integers and take: $d = 3$, $q_1 = 5$, $q_2 = 2$,

$$Q_1(y) = y + 6y^2 \text{ and } Q_2(y) = 25y + 30y^2.$$

4 Length 2 classification.

In all this section, we assume that R is a UFD.

Definition 3 Let $p_1, p_2 \in \text{qt}(R)^\times$ and $Q_1, Q_2 \in \text{qt}(R)[y]$ be such that $F(x, y) = p_2y + Q_2(p_1x + Q_1(y)) \in R[x, y]$. By definition, we have $F(x, y) \in \mathcal{R}^2(R)$. If $\deg(Q_2) \leq 0$, then $F(x, y) \in \mathcal{R}^0(R)$. If $\deg(Q_2) = 1$, then $F(x, y) \in \mathcal{R}^1(R)$. If $\deg(Q_1) \leq 0$, then $\pi F(x, y) \in \mathcal{R}^1(R)$ (recall $\pi = (y, x)$). We say that $F \in R[x, y]$ is a *rational length 2 polynomial* if $\deg(Q_1) \geq 1$ and $\deg(Q_2) \geq 2$.

Remark The first two differences between $\mathcal{R}^1(R)$ and $\mathcal{R}^2(R)$ are due to the following facts: If $p_1, p_2, p_3, p_4 \in \text{qt}(R)^\times$ and $Q_1, Q_2, Q_3, Q_4 \in \text{qt}(R)[y]$, we have:

- 1) $p_1x + Q_1(y) = p_2x + Q_2(y) \Leftrightarrow p_1 = p_2$ and $Q_1 = Q_2$ (the parameters of a polynomial in $\mathcal{R}^1(R)$ are unique).
- 2) $p_1x + Q_1(y) \in R[x, y] \Leftrightarrow p_1 \in R$ and $Q_1 \in R[y]$ ($\mathcal{R}^1(R) = \mathcal{B}^1(R)$).
- 3) $p_2y + Q_2(p_1x + Q_1(y)) = p_4y + Q_4(p_3x + Q_3(y)) \not\Leftrightarrow p_1 = p_3, p_2 = p_4$ and $Q_1 = Q_3, Q_2 = Q_4$ (the parameters of a rational length 2 polynomial are not unique).
- 4) $p_2y + Q_2(p_1x + Q_1(y)) \in R[x, y] \not\Leftrightarrow p_1, p_2 \in R$ and $Q_1, Q_2 \in R[y]$ (the parameters of a rational length 2 polynomial are not always in the ring R).

The following proposition shows that there exists rational length 2 polynomial which are coordinate but are not in the Berson classes.

Proposition 1 Let K be a field of characteristic zero. We consider $p_1 = z^2$ and $p_2 = -z^{-2}$ in $\text{qt}(K[z]) = K(z)$, $Q_1(y) = y + zy^2$ and $Q_2(y) = z^{-2}(y - zy^2)$ in $\text{qt}(K[z])[y] = K(z)[y]$. The following properties hold:

- 1) $N(x, y) = p_2y + Q_2(p_1x + Q_1(y)) = x - 2y(zx + y^2) - z(zx + y^2)^2$ is a rational length 2 polynomial,
- 2) $(N(x, y), p_1x + Q_1(y)) \in \text{GA}_2(K[z])$ (this is the Nagata automorphism),
- 3) For all $l \in \mathbb{N}$, we have $N(x, y) \notin \mathcal{B}^l(K[z])$.

Proof. 1) is trivial and 2) is classical (see [18]). For a proof of 3) see [4] Proposition 2.1.15.

The aim of the following definitions is to give some canonical parameters for a rational length 2 polynomial.

Definition 4

- a) We denote by $L_2(R)$ the set of all quadruplets (p_1, p_2, Q_1, Q_2) where $p_1, p_2 \in \text{qt}(R)^\times$ and $Q_1, Q_2 \in \text{qt}(R)[y]$ are such that $\deg(Q_1) \geq 1$, $\deg(Q_2) \geq 2$ and $p_2y + Q_2(p_1x + Q_1(y)) \in R[x, y]$.
- b) We define an equivalence relation \simeq between quadruplets in $L_2(R)$ in the following way: $(p_1, p_2, Q_1, Q_2) \simeq (p_3, p_4, Q_3, Q_4)$ if there exists $r \in R$ such that $p_2y + Q_2(p_1x + Q_1(y)) + r = p_4y + Q_4(p_3x + Q_3(y))$.
- c) A quadruplet $(p_1, p_2, Q_1, Q_2) \in L_2(R)$ is said to be *reduced* if the following conditions hold: $Q_1(0) = Q_2(0) = 0$, $p_1 \in \mathcal{U}(R)$, $Q_1(y) \in R[y]$ and $\gcd(p_1, Q_1(y)) = 1$. We denote by $L_2^{\text{red}}(R)$ the subset of all reduced quadruplets.

Proposition 2 *Every quadruplet in $L_2(R)$ is equivalent to a unique reduced quadruplet.*

Proof. Let $(p_1, p_2, Q_1, Q_2) \in L_2(R)$ be a quadruplet.

- 1) We change $Q_1(y)$ to $Q_1(y) - Q_1(0)$ and $Q_2(y)$ to $Q_2(y + Q_1(0))$.
- 2) We change $Q_2(y)$ to $Q_2(y) - Q_2(0)$.
- 3) Let $m \in R^\times$ be the smallest common multiple of the denominators of p_1 and all the coefficients of $Q_1(y)$. We change $Q_2(y)$ to $Q_2(\frac{y}{m})$, p_1 to mp_1 and $Q_1(y)$ to $mQ_1(y)$.
- 4) Let $u \in R^*$ be such that $p_1 = u w_R(p_1)$ ($w_R(p_1) \in \mathcal{U}(R)$ see Notation 1 a). We change $Q_2(y)$ to $Q_2(uy)$, p_1 to $w_R(p_1)$ and $Q_1(y)$ to $u^{-1}Q_1(y)$.

After these 4 modifications, we obtain a reduced quadruplet of $L_2^{\text{red}}(R)$ which is equivalent to (p_1, p_2, Q_1, Q_2) .

Now, let $(p_1, p_2, Q_1, Q_2), (p_3, p_4, Q_3, Q_4) \in L_2^{\text{red}}(R)$ be equivalent reduced quadruplets. There exists $r \in R$ such that $p_2y + Q_2(p_1x + Q_1(y)) + r = p_4y + Q_4(p_3x + Q_3(y))$. Taking $x = y = 0$, we obtain $r = 0$. After changing x to $p_1^{-1}x - p_1^{-1}Q_1(y)$ we have:

$$(*) \quad p_2y + Q_2(x) = p_4y + Q_4(p_3p_1^{-1}x + Q_3(y) - p_3p_1^{-1}Q_1(y)).$$

Taking $x = 0$ in $(*)$, we have: $p_2y = p_4y + Q_4(Q_3(y) - p_3p_1^{-1}Q_1(y))$. Since $\deg(Q_4) \geq 2$ and $Q_1(0) = Q_3(0) = 0$, we deduce $p_2 = p_4$ and $p_1Q_3(y) = p_3Q_1(y)$. Since $\gcd(p_1, Q_1(y)) = \gcd(p_3, Q_3(y)) = 1$ this implies $p_1 \sim p_3$ and $p_1 = p_3$ (because $p_1, p_2 \in \mathcal{U}(R)$) and $Q_1(y) = Q_3(y)$. Now $(*)$ gives $Q_2(y) = Q_4(y)$.

Proposition 3 *Let $(p_1, p_2, Q_1, Q_2) \in L_2^{\text{red}}(R)$ be a reduced quadruplet. We set $F(x, y) = p_2y + Q_2(p_1x + Q_1(y))$. Then following properties hold:*

- 1) $F(0, 0) = 0$,
- 2) $p_1p_2 \in R$,
- 3) $p_2 \in R$ if and only if $Q_2(y) \in R[y]$,

4) There exist $d \in \mathcal{U}(R)$, $q_1, q_2 \in R^\times$ and a polynomial $\tilde{Q}_2 \in R[y]$ unique such that $\gcd(d, q_2) = 1$, $p_1 = dq_1$, $p_2 = d^{-1}q_2$ and $Q_2(y) = d^{-1}\tilde{Q}_2(y)$.

Proof. 1) $F(0, 0) = Q_2(Q_1(0)) = 0$.

2) We have: $p_1p_2 = p_1\partial_y F(x, y) - Q'_1(y)\partial_x F(x, y) \in R[x, y] \cap \text{qt}(R) = R$.

3) If $Q_2(y) \in R[y]$ then $p_2y = F(x, y) - Q_2(p_1x + Q_1(y)) \in R[x, y] \cap \text{qt}(R)[y] = R[y]$ and $p_2 \in R$. Conversely, if $p_2 \in R$ then $Q_2(p_1x + Q_1(y)) \in R[x, y]$. By contradiction, let us assume $Q_2(y) \notin R[y]$. Let $m \in R^\times \setminus R^*$ be the smallest common multiple of all denominators of coefficients in $Q_2(y)$. Then $(mQ_2)(p_1x + Q_1(y)) \in mR[x, y]$ and $mQ_2 \in R[y]$ with $\gcd(mQ_2(y)) = 1$ and $p_1x + Q_1(y) \in R[x, y]$ with $\gcd(p_1x + Q_1(y)) = 1$ which is impossible.

4) There exist $d \in \mathcal{U}(R)$ and $q_2 \in R^\times$ such that $p_2 = d^{-1}q_2$ and $\gcd(d, q_2) = 1$. By 2), we have: $p_1p_2 \in R$, hence $p_1q_2 \in dR$ and this implies $p_1 \in dR$ since $\gcd(d, q_2) = 1$. In other words, there exists $q_1 \in R^\times$ such that $p_1 = dq_1$. We have: $q_2y + (dQ_2)(p_1x + Q_1(y)) = d(p_2y + Q_2(p_1x + Q_1(y))) \in R[x, y]$, hence $(p_1, q_2, Q_1, dQ_2) \in L_2^{\text{red}}(R)$. By 3), (since $q_2 \in R$), we have: $\tilde{Q}_2(y) = dQ_2(y) \in R[y]$.

Theorem 7 Assume R is PID. Then in Theorem 6, we have constructed all coordinates of rational length 2.

Proof. This result follows from 4) of Proposition 3 and Theorem 6 and the fact that $\gcd(d, q_2) = 1$ is equivalent to $dR + q_2R = R$ when R is a PID.

For all the remaining of this section, we fix $(p_1, p_2, Q_1, Q_2) \in L_2^{\text{red}}(R)$ a reduced quadruplet such that $F(x, y) = p_2y + Q_2(p_1x + Q_1(y)) \in \text{VA}_2(R)$.

Definition 5

- 1) We say that F is tame if there exists a tame automorphism σ of $R[x, y]$ such that $\sigma(y) = F$.
- 2) We say that F has a mate of length 1 if there exists $G \in \mathcal{B}^1(R)$ such that $(F, G) \in \text{GA}_2(R)$.
- 3) We say that F has length "1 + 1" if there exists $\sigma, \tau \in \text{GA}_2(R)$ such that $\sigma(y), \tau(y) \in \mathcal{B}^1(R)$ and $\sigma\tau(y) = F$.

Remark Coordinate of length "1 + 1" may be constructed with the help of Theorem 2 and may be considered as trivial from the length 2 point of view.

Theorem 8 1) F is tame if and only if there exist $\sigma, \tau \in \text{GA}_2(R)$ such that $\sigma(x), \sigma(y), \tau(x), \tau(y) \in \mathcal{B}^1(R)$ and $\sigma\tau(y) = F$.

2) F has a mate of length 1 if and only if $p_1p_2 \in R^*$.

3) F has length "1 + 1" if and only if $\phi_{p_1}(Q_1(y)) \in \text{VA}_1(R/q_1R)$.

4) if F is tame or if F has a mate of length 1 then F has length "1 + 1".

Proof.

1) We assume that F is tame. Let ρ be a tame automorphism of $R[x, y]$ such that $\rho(y) = F$. We set: $\tau_1 = (p_1x + Q_1(y))$ and $\tau_2 = (p_2x + Q_2(y))$, τ_1 and τ_2 are triangular automorphisms of $\text{qt}(R)[x, y]$. By Corollary 2, there exists τ_3 a triangular automorphism of $\text{qt}(R)[x, y]$ such that $\rho = \tau_1\pi\tau_2\pi\tau_3$. The amalgamated structure of $\text{GA}_2(\text{qt}(R))$ implies that $\rho = b_1a_1b_2a_2b_3$ where a_i (resp. b_i) are affine (resp. triangular) automorphisms of $R[x, y]$. Let b'_3 be such that $b'_3(x) = x$ and $b'_3(y) = b_3$ (b'_3 is an affine automorphism). We set: $\sigma = b_1a_2$ and $\tau = b_2a_2b'_3$. We have: $\sigma\tau(y) = b_1a_1b_2a_2b'_3(y) = b_1a_1b_2a_2b_3(y) = \rho(y) = F$ and we easily verify that $\sigma(x), \sigma(y), \tau(x), \tau(y) \in \mathcal{B}^1(R)$. Conversely, if there exist $\sigma, \tau \in \text{GA}_2(R)$ such that $\sigma(x), \sigma(y), \tau(x), \tau(y) \in \mathcal{B}^1(R)$ and $\sigma\tau(y) = F$ then σ and τ are tame automorphisms and F is tame.

2) At first, we assume that there exist $p_3 \in R^\times$ and $Q_3 \in R[y]$ such that $\sigma = (F(x, y), p_3x + Q_3(y)) \in \text{GA}_2(R)$. Composing σ with a translation we can assume that $Q_3(0) = 0$. We consider $\tau = (p_3x + Q_3(y), y) \in \text{GA}_2(\text{qt}(R))$. We have: $\pi\tau^{-1}\sigma = (p_2x + Q_2(p_1p_3^{-1}y + Q_1(x) - p_1p_3^{-1}Q_3(x)), y) \in \text{GA}_2(\text{qt}(R))$. Corollary 2 gives $p_2x + Q_2(p_1p_3^{-1}y + Q_1(x) - p_1p_3^{-1}Q_3(x)) \in \text{qt}(R)x + \text{qt}(R)[y]$. Since $\deg(Q_2) \geq 2$ and $Q_1(0) = Q_3(0) = 0$, we deduce $p_1Q_3(y) = p_3Q_1(y)$ (*) and then $p_1Q'_3(y) = p_3Q'_1(y)$ (**).

Since $\gcd(p_1, Q_1(y)) = 1$, (*) implies that there exists $u \in R$ such that $p_3 = up_1$. Since $u(p_1x + Q_1(y)) = p_3x + Q_3(y) = \sigma(y) \in \text{VA}_2(R)$, we deduce that $u \in R^*$ (when R is a domain all coordinates are irreducible polynomials). Using (**) we obtain:

$$\begin{aligned} \det(J\sigma) &= Q'_3(y)p_1Q'_2(p_1x + Q_1(y)) - p_3(p_2 + Q'_1(y)Q'_2(p_1x + Q_1(y))) \\ &= -p_2p_3 \in R^*. \end{aligned}$$

Finally $p_2p_3 = up_1p_2 \in R^*$ implies $p_1p_2 \in R^*$.

Conversely, we assume that $u = p_1p_2 \in R^*$. Changing p_1 to $u^{-1}p_1$, $Q_1(y)$ to $u^{-1}Q_1(y)$ and $Q_2(y)$ to $Q_2(uy)$, one can assume $u = 1$.

We set: $Q_3(y) = -p_1Q_2(y) \in \text{qt}(R)[y]$. We have: $Q_3(p_1x + Q_1(y)) = y - p_1F(x, y) \in R[x, y]$. Since $\gcd(p_1, Q_1(y)) = 1$, this implies $Q_3(y) \in R[y]$. From $y - Q_3(p_1x + Q_1(y)) = p_1F(x, y) \in p_1R[x, y]$, we deduce that $\phi_{p_1}(Q_1)$ and $\phi_{p_1}(Q_3)$ are inverse in $\text{GA}_1(R/p_1R)$ and $(F(x, y), p_1x + Q_1(y)) \in \text{GA}_2(R)$ by Theorem 2.

3) We assume that there exist $\sigma, \tau \in \text{GA}_2(R)$ such that $\sigma(y), \tau(y) \in \mathcal{B}^1(R)$ and $\sigma\tau(y) = F$. There exist $p_3, p_4 \in R^\times$ and $Q_3(y), Q_4(y) \in R[y]$ such that $\sigma(y) = p_3x + Q_3(y)$ and $\tau(y) = p_4x + Q_4(y)$. Let $u \in R^*$ be such that $p_3 = uw_R(p_3)$. Changing (σ, τ) to $(\sigma\rho, \rho^{-1}\tau)$ where $\rho = (x, u(y - Q_3(0)))$ we can assume that $p_3 \in \mathcal{U}(R)$ and $Q_3(0) = 0$. Since $\sigma(y) \in \text{VA}_2(R)$, Theorem 2 implies $\gcd(p_1, Q_1(y)) = 1$. By Remark 3 of Theorem 2, there exist $v \in R^*$ and $Q_5(y) \in R[y]$ such that $\sigma(x) = vp_3^{-1}(Q_5(p_3x + Q_3(y) - y))$ and $Q_5(Q_3(y)) = y \bmod p_3$. We have:

(*) $F(x, y) = \sigma\tau(y) = -vp_3^{-1}p_4y + (vp_3^{-1}p_4Q_5 + Q_4)(p_3x + Q_3(y))$.

Using (*), we prove that $(p_3, -vp_3^{-1}p_4, Q_4, vp_3^{-1}p_4Q_5 + Q_4) \in L_2^{red}(R)$ and this quadruplet is equivalent to (p_1, p_2, Q_1, Q_2) . Uniqueness in Proposition 2 gives $p_1 = p_3$ and $Q_1(y) = Q_3(y)$ and then $\phi_{p_1}(Q_1(y)) \in \text{VA}_1(R/p_1R)$ by Theorem 2. Conversely, we assume that $\phi_{p_1}(Q_1(y)) \in \text{VA}_1(R/p_1R)$. Let $Q_5(y) \in R[y]$ be such that $Q_5(Q_1(y)) = y \bmod p_1$. By Theorem 2, we have: $\sigma = (p_1^{-1}(Q_5(p_1x + Q_1(y)) - y), p_1x + Q_1(y)) \in \text{GA}_2(R)$. We have: $\sigma^{-1}(F(x, y)) = -p_1p_2x + p_2Q_5(y) + Q_2(y) \in \mathcal{B}^1(R) \cap \text{VA}_2(R)$. There exists $\tau \in \text{GA}_2(R)$ such that $\tau(y) = \sigma^{-1}(F(x, y))$ and finally $F = \sigma\tau(y)$.

4) If F is tame F then, by 1), F has length "1 + 1". If F has a mate of length 1 then there exists $\sigma \in \text{GA}_2(R)$ such that $\sigma(y) = G$ and $\sigma(x) = F$. If we set: $\tau = \pi$ then $\tau(y) = x \in \mathcal{B}^1(R)$ and $\sigma\tau(y) = \sigma(x) = F$, and F has length "1 + 1".

Remark. Let us assume that $R = K[z]$ where K is a field of characteristic 0. Let F be rational length 2 coordinate. Using 1) of Theorem 8 one can check if F is tame coordinate of $K[z][x, y]$. If is not, then Theorem 4 implies that F is a wild coordinate of $K[x, y, z]$.

5 Equivalent polynomials.

In this section, we assume that R is a UFD.

Definition 6 Let $F, G \in R[x, y]$ we say that F and G are *equivalent* if there exists $\sigma \in \text{GA}_2(R)$ such that $\sigma(F) = G$. This is of course a equivalent relation.

Theorem 9 Let $p_1, p_2 \in R^\times$ be nonzero elements and $Q_1, Q_2 \in R[y]$ be polynomials such that $\gcd(p_1, Q_1(y)) = \gcd(p_2, Q_2(y)) = 1$ and $Q_1(0) = Q_2(0) = 0$.

1) The polynomials $p_1x + Q_1(y)$ and $p_2x + Q_2(y)$ (in $\mathcal{B}^1(R)$) are equivalent if and only if there exist $Q_3 \in R[y]$ and $u \in R^*$ such that:

$$(*) \quad Y = u \frac{\gcd(p_1, p_2)}{p_2} \left\{ \frac{p_1}{\gcd(p_1, p_2)} y + Q_3(p_2x + Q_2(y)) \right\} \in R[x, y]$$

and (**) $Q_1(Y) = p_2x + Q_2(y)$ modulo $p_1R[x, y]$.

2) If the polynomials $p_1x + Q_1(y)$ and $p_2x + Q_2(y)$ are equivalent then $p'_1x + Q_1(y)$ and $p'_2x + Q_2(y)$ are in $\text{VA}_2(A)$ where $p'_1 = p_1/\gcd(p_1, p_2)$ and $p'_2 = p_2/\gcd(p_1, p_2)$.

Proof. 1) We assume that there exists $\sigma \in \text{GA}_2(R)$ such that $\sigma(p_1x + Q_1(y)) = p_2x + Q_2(y)$. Let $\tau_1 = (p_1x + Q_1(y), y)$ and $\tau_2 = (p_2x + Q_2(y), y)$ be two triangular automorphisms of $GA_2(\text{qt}(A))$. We have: $\sigma\tau_1(x) = \tau_2(x)$ i. e. $\sigma\tau_1\pi(y) = \tau_2\pi(y)$ (recall that $\pi = (y, x)$). By Corollary 2, there exist $p_3 \in \text{qt}(R)^*$ and $Q_4 \in \text{qt}(R)[y]$ such that $\pi\tau_2^{-1}\sigma\tau_1\pi = \tau_3 = (p_3x + Q_4(y), y)$. We set $u = p_2p_3p_1^{-1}$. Since $\sigma = \tau_2\pi\tau_3\pi\tau_1^{-1} \in \text{GA}_2(R)$, we have: $u = \det(J\sigma) \in R^*$. We set: $Y = \sigma(y) = \tau_2\pi\tau_3(x) = up_1p_2^{-1}y + Q_4(p_2x + Q_2(y)) \in R[x, y]$. Since $\gcd(p_2, Q_2(y)) = 1$, this implies that there exists $Q_3 \in R[y]$ such that $Q_4(y) = u \gcd(p_1, p_2)p_2^{-1}Q_3(y)$ and (*) follows.

Finally, $\sigma(x) = \tau_2\pi\tau_3\pi(p_1^{-1}(x - Q_1(y))) = p_1^{-1}(\tau_2\pi\tau_3(y) - Q_1(\sigma(y)))$
 $= p_1^{-1}(p_2x + Q_2(y) - Q_1(Y)) \in R[x, y]$ and we obtain (**).

Conversely, if we have (*) and (**), we define an endomorphism σ of $R[x, y]$ by $\sigma(y) = Y$ and $\sigma(x) = p_1^{-1}(p_2x + Q_2(y) - Q_1(Y))$. We can check easily that $\sigma \in \text{GA}_2(\text{qt}(R))$ and $\det(J\sigma) = u \in R^*$. Lemma 3 implies that $\sigma \in \text{GA}_2(R)$ and a straight forward computation shows that $\sigma(p_1x + Q_1(y)) = p_2x + Q_2(y)$.

2) The assumption $Y = \frac{u}{p_2'}\{p_1'y + Q_3(p_2x + Q_2(y))\} \in R[x, y]$ is equivalent to $p_1'y + Q_3(Q_2(y)) \in p_2'R[y]$. Since p_1' is invertible modulo p_2' there exists such Q_3 if and only if Q_2 is invertible (for composition) modulo p_2' . By Theorem 2, this is equivalent to $p_2'x + Q_2(y) \in \text{VA}_2(R)$. By symmetry, we have also $p_1'x + Q_1(y) \in \text{VA}_2(R)$.

Remark Let σ be the automorphism in Theorem 9. Then $Y = \sigma(y)$ is a rational length 2 coordinate.

Corollary 3 Let $p_1, p_2 \in R^\times$ be such that $\gcd(p_1, p_2) = 1$ and $Q_1, Q_2 \in R[y]$ be polynomials such that $\gcd(p, Q_1(y)) = \gcd(p, Q_2(y)) = 1$ and $Q_1(0) = Q_2(0) = 0$. Then $p_1x + Q_1(y)$ and $p_2x + Q_2(y)$ are equivalent if and only both are in $\text{VA}_2(R)$.

Proof. If $p_1x + Q_1(y)$ and $p_2x + Q_2(y)$ are in $\text{VA}_2(R)$, both are equivalent to x , hence are equivalent. The converse follows from 2) of Theorem 9 (since $p_1 = p_1'$ and $p_2 = p_2'$).

Corollary 4 Let $p \in R^\times$ be a nonzero element and $Q_1, Q_2 \in R[y]$ be polynomials such that $\gcd(p, Q_1(y)) = \gcd(p, Q_2(y)) = 1$ and $Q_1(0) = Q_2(0) = 0$. There exists $\sigma \in \text{GA}_2(R)$ such that $\sigma(px + Q_1(y)) = px + Q_2(y)$ if and only if there exist $Q_3 \in R[y]$ and $u \in R^*$ such that:

$$Q_1(u\{y + Q_3(Q_2(y))\}) = Q_2(y) \mod pR[y].$$

Proof. Take $p_1 = p_2 = p$ in 1) of Theorem 9.

Example (Poloni). We consider, in the ring $R = \mathbb{C}[z]$, the element $p = z^2$, and the polynomials $Q_1(y) = -y^2 - zq_1(y)$ and $Q_2(y) = -y^2 - zq_2(y)$ where $q_1, q_2 \in \mathbb{C}[y]$ are such $q_1(0) = q_2(0) = 0$. We have the following characterization:

There exist $\sigma \in \text{GA}_2(R)$ such that $\sigma(px + Q_1(y)) = px + Q_2(y)$ if and only if $q_2(y) + q_2(-y) = q_1(y) + q_1(-y)$.

If we compose the automorphism σ with $(x, y, az) \in \text{GA}_3(\mathbb{C})$ where $a \in \mathbb{C}^*$, we obtain the if part of Theorem 4.2.28 p. 96 in [19].

In fact, using Corollary 4, there exists $\sigma \in \text{GA}_2(R)$ such that $\sigma(px + Q_1(y)) = px + Q_2(y)$ if and only if there exist $u \in R^*$ and $Q_3 \in R[y]$ such that:

$$(\dagger) u^2(y + Q_3(-y^2 - zq_2(y)))^2 + zq_1(u(y + Q_3(-y^2))) = y^2 + zq_2(y) \bmod z^2.$$

Looking at equation (\dagger) modulo z , we have:

$u^2(y^2 + 2yQ_3(-y^2) + Q_3(-y^2)^2) = y^2 \bmod z$, and we deduce: $u^2 = 1$ and $Q_3(y) = 0 \bmod z$. We can write $Q_3(y) = zQ_4(y)$ with $Q_4 \in \mathbb{C}[z][y]$. Now, (\dagger) is equivalent to $2yQ_4(-y^2) = q_2(y) - q_1(uy)$. There exists such a Q_4 if and only if $q_2(y) - q_1(uy)$ is an odd polynomial. We conclude by observing that there exist $u \in \{-1, 1\}$ such that $q_2(y) - q_1(uy) = -q_2(-y) + q_1(-uy)$ if and only if $q_1(y) + q_1(-y) = q_2(y) + q_2(-y)$.

6 Co-tame automorphisms.

In this section K is a field of characteristic 0.

We denote by $G = \text{GA}_3(K)$ the group of all automorphisms of the K -algebra $K[x, y, z]$, $A = \text{Aff}_3(K)$ the affine automorphisms sub-group, $B = \text{BA}_3(K)$ the triangular automorphisms sub-group and $T = \langle A, B \rangle_G = \text{TA}_3(K)$ the tame automorphisms sub-group.

Definition 7 Let $\sigma \in G$ we say that σ is *co-tame* if $T \subset \langle A, \sigma \rangle_G$. In other words, σ is co-tame, if every tame automorphism is in the sub-group generated by σ and all affine automorphisms.

Remark Let $\sigma \in G$ be an automorphism.

- 1) If σ is tame, we have $T \supset \langle A, \sigma \rangle_G$ (this is the origin of our terminology).
- 2) If σ is both tame and co-tame, we have $T = \langle A, \sigma \rangle_G$.
- 3) If σ is affine then σ is tame but is not co-tame.
- 4) If σ is co-tame then all automorphisms in $A\sigma A$ are also co-tame. "To be tame" and "to be co-tame" are properties of the orbits $A\sigma A$ ($\sigma \in G$).
- 5) Let $\alpha \in A$ be an affine automorphism. If $\sigma\alpha\sigma^{-1}$ is co-tame then σ is co-tame. We often use this with $\alpha = t = (x + 1, y, z)$.

Derksen first proved the existence of a co-tame automorphism (see [12] and Lemma 4). Bodnarchuk proved that a large class of tame automorphisms are co-tame (see [2] and Theorem 10). They both work in dimension $n \geq 3$ but here we focus in dimension 3.

We denote by $P = \{\sigma \in G; \sigma(y), \sigma(z) \in K[y, z]\} \subset T$ the set of *parabolic* automorphisms. The automorphisms in BAB (resp. PAP) are called bi-triangular (resp. bi-parabolic).

Lemma 4 (Derksen, 1997) *The automorphism $(x + y^2, y, z)$ is co-tame.*

Lemma 5 (Bodnarchuk, 2004) *If $\sigma \in (B \cup BAB) \setminus A$ then σ is co-tame.*

Theorem 10 (Bodnarchuk, 2004) *If $\sigma \in (P \cup PAP) \setminus A$ then σ is co-tame.*

Remark Bodnarchuk considers only tame automorphisms. He asks the following question: Is all non affine tame automorphisms are co-tame? In other words, is A is a maximal sub-group of T ? This question is still open.

Theorem 11 and Theorem 12 give a lot of examples of non tame automorphisms which are co-tame.

Theorem 11 *Let $\sigma \in G \setminus A$ be a non affine automorphism. We assume that $\sigma(z) = z$ and $\sigma(y) \in \mathcal{R}_1(K[z])$. then σ is co-tame.*

Proof Let $p_1 \in K[z]^\times$ and $Q_1(y) \in K[z][y]$ be such that $\sigma(y) = p_1x + Q_1(y)$. Since $\sigma(z) = z$ we can consider σ as an automorphism of $K(z)[x, y]$. By Theorem 2 Remark 3), we have:

$$\sigma = ((up_1)^{-1}(y - Q_2(p_1x + Q_1(y))), p_1x + Q_1(y))$$

where $u \in K^*$ and $Q_2(y) \in K[z][y]$ is such $Q_2(Q_1(y)) = y \bmod p_1K[z][y]$.

We can remark that if we consider σ as an automorphism of $K(z)[x, y]$, we have $\sigma = b_1\pi b_2^{-1}$ with $b_1 = (p_1x + Q_1(y), y)$ and $b_2 = (up_1x + Q_2(y), y)$.

Let $t = (x + 1, y, z) \in A$ be the unitary translation on x . An elementary computation gives: $\tau = \sigma t \sigma^{-1} = (x + p_1^{-1}(Q_1(y) - Q_1(y + up_1)), y + up_1, z) \in B$. If $\deg_y(Q_1) \geq 3$ then $\deg_y(Q_1(y) - Q_1(y + up_1)) \geq 2$ and $\tau \in B \setminus A$. By Lemma 5, τ and then σ are co-tame.

We assume, now, $\deg_y(Q_1) \leq 2$. We write $Q_1(y) = a + by + cy^2$ where $a, b, c \in K[z]$. We have $p_1^{-1}(Q_1(y) - Q_1(y + up_1)) = -u(2cy + b + up_1)$.

If $c \in K[z] \setminus K$ then $\tau \in B \setminus A$ and we can conclude as above.

We assume, now, $c \in K$. Since $\sigma(y) \in \text{VA}_2(K[z])$, Theorem 2 implies that

$p_1 \in K^*$ or $c = 0$. In both cases $\sigma \in BAB$ and we can conclude using Lemma 5.

Remark Using Theorem 3 and Theorem 11, we deduce that the Nagata automorphism is non tame but co-tame. The sub-group generated by Nagata automorphism and affine automorphisms strictly contains the tame automorphisms group! We don't know if this group is a proper sub-group of $\text{GA}_3(K)$.

Theorem 12 *Let $\sigma \in G \setminus A$ be a non affine automorphism. We assume that $\sigma(z) = z$ and $\sigma(y) \in \mathcal{R}_2(K[z])$. then σ is co-tame.*

Proof There exist $p_1, p_2, p_3 \in K(z)^*$ and $Q_1, Q_2, Q_3 \in K(z)[y]$ such that such that $\sigma = \tau_1 \pi \tau_2 \pi \tau_3$ where $\tau_i = (p_i x + Q_i(y), y) \in \text{BA}_2(K(z))$ for $i \in \{1, 2, 3\}$ (see Remark 1) after Definition 2). We prove that σ is co-tame by induction on $\deg_y(Q_2)$.

If $\deg_y(Q_2) \leq 1$ then $\pi \tau_2 \pi \in \text{Aff}_2(K(z))$ and the Bruhat decomposition implies that $\sigma(y) \in \mathcal{R}_1(K[z])$ and σ is co-tame by Theorem 11.

If $\deg_y(Q_2) \geq 2$ we compute $\tau = \sigma t \sigma^{-1}$ where $t = (x + 1, y, z) \in A$ is again the unitary translation on x . We have $\tau = \sigma t \sigma^{-1} = \tau_1 \pi \tau_4 \pi \tau_1^{-1}$ where $\tau_4 = \tau_2 \pi \tau_3 \pi \tau_3^{-1} \pi \tau_2^{-1} = (x + p_2^{-1}(Q_2(y) - Q_2(y + p_3^{-1})), y + p_3^{-1}) \in \text{BA}_2(K(z))$. Since $\deg_y(Q_2(y) - Q_2(y + p_3^{-1})) < \deg_y(Q_2(y))$, by induction τ and then σ are co-tame.

References

- [1] V. V. Bavula, The inversion formulae for automorphisms of polynomial algebras and rings of differential operators in prime characteristic. *J. Pure Appl. Algebra* 212 (2008), no. 10, 2320–2337.
- [2] Y. Bodnarchuk, On generators of the tame invertible polynomial maps group, *Internat. J. Algebra Comput.* 15 (2005), no. 5-6, 851867.
- [3] J. Berson, Stable tame coordinates, *J. Pure and Applied Algebra*, 170 (2002), 131-143.
- [4] J. Berson, Polynomial coordinates and their behavior in higher dimension, PhD thesis, University of Nijmegen, Nijmegen (2004).
- [5] J. Berson, A. van den Essen, S. Maubach, Derivations having divergence zero on $R[X, Y]$, *Israel J. Math.* 124 (2001), 115–124.
- [6] S. Bhatwadekar, A. Dutta, Kernel of locally nilpotent R -derivations of $R[X, Y]$, *Trans. Amer. Math. Soc.* 349 (1997), no. 8, 3303–3319.

- [7] D. Daigle, G. Freudenburg, Locally nilpotent derivations over a UFD and an application to rank two locally nilpotent derivations of $k[X_1, \dots, X_n]$, *J. Algebra* 204 (1998), no. 2, 353–371.
- [8] V. Drensky and J.-T. Yu, Tame and wild Coordinates of $K[z][x; y]$, *Trans. Amer. Math. Soc.*, 353 (2001), 519–537.
- [9] E. Edo, Totally stably tame variables, *J. Algebra* 287 (2005) 15–31.
- [10] E. Edo, J.-P. Furter, Some families of polynomial automorphisms, *J. Pure Appl. Algebra* 194 (2004), no. 3, 263–271.
- [11] E. Edo and S. Vénéreau, Length 2 variables and transfer, *Annales polonici Math.* 76 (2001), 67–76.
- [12] A. van den Essen, Polynomial Automorphisms and the Jacobian Conjecture, *Progress in Mathematics*, vol. 190, Birkhäuser-Verlag, (2000).
- [13] S. Friedland, J. Milnor. Dynamical properties of plane polynomial automorphisms, *Ergod. Th. Dyn. Syst.* 9 (1989), 67–99.
- [14] J-P. Furter, On the variety of automorphisms of the affine plane. *J. Algebra* 195 (1997), no. 2, 604–623.
- [15] H. Jung Über ganze birationale Transformationen der Ebene. *J. Reine Angew. Math.* 184 (1942). 161–174.
- [16] W. van der Kulk. On polynomial rings in two variables. *Nieuw Arch. Wiskunde* (3) 1 (1953), 33–41.
- [17] A. Mikhalev, V. Shpilrain and J.-T. Yu, Combinatorial methods. Free groups, polynomials, and free algebras. CMS Books in Mathematics 19. Springer-Verlag, New York, 2004. ISBN: 0-387-40562-3.
- [18] M. Nagata, On the automorphism group of $k[X, Y]$, *Kyoto Univ. Lectures in Math.* 5 (1972).
- [19] P.-M. Poloni. Sur les plongements des hypersurfaces de Danielewski. Thèse de doctorat. Université de Bourgogne (2008).
- [20] P. Russell. Simple birational extensions of two dimensional affine rational domains. *Compositio Math.* 33 (1976), no. 2, 197–208.
- [21] I. Shestakov and U. Umirbaev, The tame and wild automorphisms of polynomial rings in three variables, *J. of the AMS*, 17 (2004), no. 1, 197–227.
- [22] U. Umirbaev and J.-T. Yu, The strong Nagata conjecture. *Proc. Natl. Acad. Sci. USA*, 101 (2004), no. 13.

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